The Generic Multiple-Precision Floating-Point Addition With Exact Rounding (as in the MPFR Library)

Vincent Lefèvre

INRIA Lorraine, 615 rue du Jardin Botanique, 54602 Villers-lès-Nancy Cedex, France

Abstract

We study the multiple-precision addition of two positive floating-point numbers in base 2, with exact rounding, as specified in the MPFR library, i.e. where each number has its own precision. We show how the best possible complexity (up to a constant factor that depends on the implementation) can be obtain.

Key words: multiple precision, floating point, addition, exact rounding

1 Introduction

In this paper, we consider the multiple-precision floating-point addition with exact rounding, as specified in the MPFR library [1] : the inputs are two (binary) floating-point numbers x and y of precision $m \geq 2$ and $n \geq 2$, a target precision $p \geq 2$ and a rounding mode $\diamond$, and the output is $\diamond(x+y)$, i.e. the exact value $x+y$ rounded to the target precision in the given rounding mode, and a ternary value giving the sign of $\diamond(x+y) - (x+y)$.

By "addition", we mean here the "true" addition, that is $x+y$ where x and y have the same sign, and $x-y$ where x and y have opposite signs. For the sake of simplicity, we restrict to the addition of positive values for x and y in the following of the paper. In fact, this is how it is implemented in MPFR: indeed, the addition and subtraction functions call an auxiliary function, ignoring the signs of the input numbers (they are regarded as positive).

Email address: `Vincent.Lefevre@loria.fr` (Vincent Lefèvre).
URL: `http://www.vinc17.org/research/` (Vincent Lefèvre).

[1] http://www.mpfr.org/

The addition seems to be a very simple function to implement, as being a basic function, easy to understand. This is unfortunately not true, in particular under the MPFR specifications (where the inputs and the output may have different precisions and the output must be the exactly rounded result), because many different cases need to be considered, and it is easy to forget one, both in the implementation of the addition and in the tests. For a long time, the MPFR addition had been buggy (more precisely, some rare special cases were not handled correctly) and inefficient (also in some rare special cases, for which the time complexity was exponential). I completely rewrote the addition in October 2001. The presentation given in this paper is more or less based on the same ideas as the ones used in the new MPFR implementation, with some non-theoretical differences, mainly due to some MPFR internals.

First, the floating-point system is introduced in Section 2. The main computation steps (from which the algorithm can be deduced) and the complexity are presented in Section 3. Section 4 deals with the MPFR implementation. We finally conclude in Section 5.

2 The Floating-Point System

2.1 The Floating-Point Representation

We consider a floating-point system in base 2. The results presented in this paper can naturally be extended to other fixed even bases, but we choose the base 2 (this is the base of the floating-point system in MPFR) to make the notations easier to understand.

In our system, an object may contain a special value, like NaN (not a number) or an infinity, zero (possibly signed, as in the IEEE-754 standard[1] and in MPFR), or an non-zero real number that can be written:

$$s \times 0.b_1 b_2 b_3 \ldots b_p \times 2^e,$$

where $s = \pm 1$ is the *sign*, the b_i's are binary digits (0 or 1) forming the *mantissa*, e is the *exponent* (a bounded integer [2]), and p is an integer greater or equal to 1, called the *precision*. In MPFR, the precision p is not fixed; it is attached to each object. Non-zero real values are normalized, i.e. $b_1 \neq 0$, that is in base 2, $b_1 = 1$. The system does not have subnormals (i.e. numbers with $b_1 = 0$) as they are not really useful with a huge exponent range (like in MPFR) and would make the algorithms and the code much more complex.

[2] In MPFR, the exponent is between $1 - 2^{30}$ and $2^{30} - 1$.

In the following of the paper, we consider only positive input numbers (as said in the introduction), i.e. numbers that can be written: $0.1b_2b_3\ldots b_p \times 2^e$.

2.2 Rounding

When adding two positive numbers x and y of respective precisions m and n, the result is not necessarily representable in the target precision p; it must be rounded, according to one of the rounding modes chosen by the user, similar to the IEEE-754 rounding modes:

- rounding to minus infinity (downwards): we return the largest floating-point number in precision p that is less or equal to $x+y$;
- rounding to plus infinity (upwards): we return the smallest floating-point number in precision p that is greater or equal to $x+y$;
- rounding towards zero: we round downwards, since $x+y>0$;
- rounding to the nearest: we return the floating-point number in precision p that is the closest to $x+y$. Halfway cases are specified by the implementation; if $p \geq 2$ (this is required by MPFR), then we can choose the round-to-even rule, like in the IEEE-754 standard and in MPFR: we return the only number that has an even mantissa, i.e. with $b_p = 0$.

Note that the returned result must be the rounding of the exact result; this requirement is called *correct* or *exact rounding*.

In addition to the rounded result, a ternary value is returned, giving the sign of $\diamond(x+y)-(x+y)$, where $\diamond$ denotes the chosen rounding mode: a positive number means that the rounded result is greater or equal to the exact result, a negative number means that the rounded result is less or equal to the exact result, and 0 means that the returned result is the exact result.

Moreover, it is possible that the exponent of the rounded result is not in the exponent range, in which case an overflow is generated. This case does lot lead to any practical or theoretical difficulty and is beyond the scope of this paper.

How the exact result of a canonical infinite mantissa $0.1b_2b_3\ldots$ (where the number of zero bits is infinite) is rounded can be expressed as a function of the bit $r = b_{p+1}$ following the truncated p-bit mantissa, called the *rounding bit*, and $s = b_{p+2} \vee b_{p+3} \vee \ldots$, called the *sticky bit*, as summarized in Table 1 (we recall that the result is positive).

Note: We did not mention the ternary value, as it can easily be deduced from Table 1 (telling how the mantissa is rounded). Also, like the rounding modes towards $-\infty$ and towards $+\infty$, the returned ternary value needs to be negated if the result is negative (not considered in this paper).

r / s	downwards	upwards	to the nearest
0 / 0	exact	exact	exact
0 / 1	$-$	$+$	$-$
1 / 0	$-$	$+$	$-$ / $+$
1 / 1	$-$	$+$	$+$

Table 1
A $-$ in the table means that the mantissa of the exactly rounded result is $0.1b_2b_3\ldots b_p$, i.e. the truncated exact mantissa. A $+$ in the table means that one needs to add 2^{-p} to the truncated mantissa (leading to an exponent change if all the b_i's up to b_p are 1). The $-$ / $+$ corresponds to the halfway cases, and the round-to-even rule is applied, that is: $-$ if $b_p = 0$, $+$ if $b_p = 1$.

3 The Main Computation Steps and the Complexity

We still denote the precisions of the input numbers x and y and the result by m, n and p respectively.

The addition of two positive floating-point numbers x and y of respective exponents e_x and e_y consists in:

(1) ordering x and y so that $e_x \geq e_y$,
(2) computing the exponent difference $d = e_x - e_y$,
(3) shifting the mantissa of y by d positions to the right,
(4) adding the mantissa of x and the shifted mantissa of y and rounding the result (shifting it by 1 position to the right if there is a carry),
(5) computing the exponent of the result: e_x or $e_x + 1$ if there is a carry.

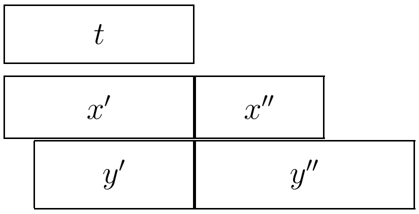
This method is very inefficient if many trailing bits of x or y (possibly all the bits of y) do not have any influence on the result, for instance:

$$0.101010000010010001 + 0.10001 \times 2^{-9}$$

rounded to 4 bits. The exactly rounded result and the ternary value can be deduced from only the first 6 bits 101010 of x (and none for y), knowing the fact that its first mantissa bit is always 1.

So, we are interested in taking into account as few input bits as possible (the possible *hole* between the least significant bit of x and the most significant bit of y must also be detected). We do not have any particular knowledge about the input numbers x and y (and the result); we assume that the mantissa bits are 0 and 1 with equal probabilities after some given position and that x and y are independent numbers. Of course, this is not necessarily a good assumption, but this will be discussed when it has an importance.

The addition can be written $x+y=t+\varepsilon$, where t is the *main term*, computed with the first $p+2$ bits of x and the corresponding $\max(p+2-d,0)$ bits of y, and ε is the *error term*, satisfying $0 \le \varepsilon < 2^{e_x-p-1}$. This can graphically be represented by:



where x'' may be empty and either y' or y'' may be empty.

The *main term* t is computed and written in time $\Theta(p)$; indeed, an $\Omega(p)$ time is necessary to fill the $p+2$ bits, and a linear time is obviously sufficient. There are many ways to deal with all the different cases (the mantissas of x and y may completely overlap, partially overlap in numerous ways, or even not overlap at all, and some parts of the result may need to be filled with zeros); a carry detection can also be performed by looking at the most significant bits of x and y first. More will be said in Section 4, about the implementation in MPFR. However this is not an important point here, as long as the complexity is in $\Theta(p)$.

The *error term* allows to obtain the truncated mantissa, the rounding bit and the sticky bit (Section 2.2). First, if the computation of the main term has lead to a carry, then $p+3$ bits of the result have really been computed. This case can be regarded as if there were no carry and the first iteration of the processing described below were already performed (then, this is only a matter of implementation). So, for the sake of simplicity, let us consider that $p+2$ bits of the result have been computed, let u denote the *weight* [3] of the bit $p+2$ (so, $0 \le \varepsilon < 2u$), and let f denote its value (0 or 1) [4], that we call the *following* bit. Table 2 gives the rounding bit r and the sticky bit s as a function of the following bit f and the error ε.

Combining Tables 1 and 2, we get Table 3. Now we may need to determine the sign of $\varepsilon - fu$ (depending on the cases given by Table 3). This is done with an iteration over the remaining bits of x and y.

- If $f=0$, we need to distinguish the cases $\varepsilon = 0$ and $\varepsilon > 0$. We have: $\varepsilon > 0$ if and only if at least a trailing bit (of x or y) is 1. In particular, if $y < u$,

[3] This is the corresponding power of 2; for instance, the weight of the bit 1 in 0.001 is 2^{-3}.

[4] If a carry was generated, consider only the first $p+2$ bits of the result in t, and the bit $p+3$ is taken into account in ε.

f	ε	r	s
0	$\varepsilon = 0$	$=$	0
0	$\varepsilon > 0$	$=$	1
1	$\varepsilon < u$	$=$	1
1	$\varepsilon = u$	$+$	0
1	$\varepsilon > u$	$+$	1

Table 2
For r, an $=$ means that the rounding bit is the bit $p + 1$ of the temporary result t, and a $+$ means that 1 must be added to the bit $p + 1$ of t (and the carry must propagate).

r_t	f	ε	r	s	downwards	upwards	to the nearest
0	0	$\varepsilon = 0$	0	0	exact	exact	exact
0	0	$\varepsilon > 0$	0	1	$-$	$+$	$-$
0	1	$\varepsilon < u$	0	1	$-$	$+$	$-$
0	1	$\varepsilon = u$	1	0	$-$	$+$	$-$ / $+$
0	1	$\varepsilon > u$	1	1	$-$	$+$	$+$
1	0	$\varepsilon = 0$	1	0	$-$	$+$	$-$ / $+$
1	0	$\varepsilon > 0$	1	1	$-$	$+$	$+$
1	1	$\varepsilon < u$	1	1	$-$	$+$	$+$
1	1	$\varepsilon = u$	0	0	exact	exact	exact
1	1	$\varepsilon > u$	0	1	$-$	$+$	$-$

Table 3
The first three columns give all the possible cases for the rounding bit of the main term, the following bit f and the error ε. The next two columns give the corresponding values of the rounding bit r and the sticky bit s (once the error has been taken into account). The last three columns give information for the rounded result and the ternary value; in the last two cases (lines), a carry is added to the mantissa before the rounding (and this may lead to an exponent change, but has no effect on how the rounding is performed — implementations must just take care that the ulp is different when rounding upwards).

then the most significant bit of y (always 1) is a trailing bit (said otherwise, $\varepsilon \geq y$); so, in this case, $\varepsilon > 0$. Otherwise, one needs to test the trailing bits, the one after the other until a 1 is found, and in the worst case ($\varepsilon = 0$), all the trailing bits need to be tested. As a consequence, the worst-case complexity in the case $f = 0$ is in $\Theta(m + n + p)$.

Is there a best order to test the trailing bits? Under the condition that we do not have any particular knowledge on the input numbers, there is no best order[5]. One can choose one of the following two possibilities, for instance:

· Test the trailing bits of x, then the trailing bits of y (or the other way round) until a non-zero bit is found.
· Test trailing bits of both x and y at the same time. This may be an interesting choice as some numbers tend to have an exact mantissa with few non-zero digits (like small integers), thus many trailing zeros. Testing trailing bits of x and y concurrently may allow to avoid such difficult cases.

- If $f = 1$, we need to distinguish the cases $\varepsilon < u$, $\varepsilon = u$ and $\varepsilon > u$. Let d denote the exponent difference so that the bits x_i and y_{i-d} have the same weight (d is the shift count to align the mantissas). Let q be the first integer such that the trailing bits x_q and y_{q-d} are equal (when a bit is not represented, it is 0). If these bits are 0's, then $\varepsilon < u$. Otherwise (i.e. if these bits are 1's), $\varepsilon \geq u$. The equality $\varepsilon = u$ can be decided as in the case $f = 0$ ($\varepsilon > u$ if and only if at least one the untested bits is 1).

  The best way is to start with bit $p + 3$ and loop over the increasing positions, until q is found (and more if one has to decide if $\varepsilon = u$). If the x mantissa or the y mantissa has entirely been read and q has not been found yet, then it is not necessary to go further, as we can deduce that $\varepsilon < u$; in other words, there is no possible carry with bits from only one mantissa.

  Up to the position $q - 1$, we have $x_i + y_{i-d} = 1$, as one of the bits is 0 and the other is 1. When x and y overlap, it is necessary to test at least one bit (as one of these two bits has an importance in the sign of $\varepsilon - u$). Concerning the other bits, the result can be deduced from only one test (like in the case $f = 0$), but an oracle would be needed, and it is not possible to do better without any particular knowledge. Here is an example: let us considered $x = 0.101111100101$ (12-bit precision), $y = 0.11010 \times 2^{-7}$ (5-bit precision), and a 2-bit target precision. The mantissas are aligned in the following way:

$$\begin{array}{rl} & 0.101111100101 \\ + & 0.11010 \end{array}$$

  Though on this particular example, testing the last five bits 0 is sufficient to deduce that the exact result is less than 0.11, all the bits are tested from the left to the right. Moreover, if either x or y (but not both) had more bits, e.g. $y = 0.11010111001 \times 2^{-7}$, then testing these bits would not be necessary as they cannot generate a carry to reach 0.11; however, if

[5] One can argue that the real numbers naturally are logarithmically distributed, so that the probability to have a 0 at position i is higher than the one to have a 1, and the difference decreases as a function of i [2]. Therefore, in a very theoretical point of view, if the time of each test is seen as a constant, it would be better to start by the least significant bits! Of course, since the probabilities get very close to 1/2 very quickly, one would not see any difference in practice.

$y = 0.110110000 \times 2^{-7}$, testing the following four bits would be necessary to deduce that the result is 0.11 exactly.

In the case $f = 1$, the loop is performed over the increasing positions from the bit $p + 3$, grabbing the bit of x and the bit of y having the same weight. The same loop can be performed in the case $f = 0$, though this is not the only solution as said above. Of course, special cases must be taken into account: the x mantissa does not necessarily overlap with the aligned y mantissa (as the most significant bit of y may come after some trailing bits of x, some trailing bits of x may come after the least significant bit of y, some trailing bits of y may come after the least significant bit of x), and the most significant bit of y may come after the least significant bit of x (*hole* between the x mantissa and the y mantissa), any *missing* bit being regarded as 0. At each iteration, the mantissa of the temporary result has the form: $0.1z_2z_3 \ldots z_p r f f f \ldots f f f$ with an error in the interval $[0, 2)$ ulp[6]. One iterates as long as the bits after the (temporary) rounding bit are identical. This basically corresponds to the Table Maker's Dilemma, that occurs to exactly round any function (see [3], for instance).

The time complexity is in $\Omega(p)$ and in $O(m+n+p)$. In the worst case, it is in $\Theta(m + n + p)$. In average (if the bits are 0 and 1 with equal probabilities and input numbers are independent), the complexity is in $\Theta(p)$, as the probability to need to test k trailing bits decreases exponentially (in 2^{-k}).

4 The MPFR Implementation

In this section, we present the implementation of the addition in MPFR. To keep the paper from being too technical, we do not give many details (and the reader can read the source code, as MPFR is distributed under the GNU Lesser General Public License). A complete proof would also be very hard to read and check (unless it could be mechanically checked); so, such a proof is not provided.

Let us start with the representation of MPFR numbers. Non-special MPFR numbers have a sign (accessed with C macros), a mantissa, an exponent (some C integer) and a precision (also some C integer). The mantissa is represented by an array of *limbs*; a limb is an unsigned integer (having 32 bits or 64 bits, depending on the C implementation), as defined in the GMP library[7], on which MPFR is based. All the bits of a limb are used to represent bits

[6] Unit in the Last Place: here, the weight of the last bit f of the temporary result.
[7] http://www.swox.com/gmp/

of the mantissa in the conventional binary representation [8]. The mantissa is normalized, i.e. its most significant bit is always 1. Since the precision is not necessarily a multiple of the limb size, some bits of the lowest mantissa limb are not significant and are always 0 (except in temporary values).

The computation steps presented in Section 3 were bit based (as this is more regular and easier to understand for a theoretical analysis). But working on single bits in a software implementation would not be very efficient. Base operations must be performed by blocks; some limbs may still need to be split into two parts as the y mantissa must be aligned with the x mantissa.

In addition to the particular cases that arise in the bit-based case, we need to distinguish the case where the exponent difference is a multiple of the limb size and the other case, needing the y mantissa to be shifted (this is usually done on the fly). We also need to take into account all the cases related to the block boundaries (for instance, where the rounding bit lies in a limb).

First, the main term is computed, but there are differences with the bit-based version. As the array holding the target mantissa does not necessarily have the room for $p+2$ bits and we want to avoid an inefficient memory allocation for $p+2$ bits and copy, the temporary rounding bit and the following bit are stored in C integer variables `rb` and `fb` (determined on the fly, as soon as they are known); in this way, then can also be handled more efficiently. The second difference is all the bits of the target array are used for this computation (in fact, this is more or less necessary, as the low-level GMP functions do not perform any masking).

For the main term, we want to add the most significant parts x' and y'. If y does not overlap with the main term (y' is empty), we just copy x' to the target array and zero the least significant limbs of the target if the target has a greater precision than x. Now, let us assume that y overlaps with the main term. With GMP, we cannot shift and add with a single operation; therefore these operations have to be performed separately. First, with a GMP function, we copy the most significant part of y, shifted if need be, to the target array and we zero the limbs of the target that have not been touched: the most and/or least significant limbs, if the exponent and the precision of y are small enough. Then, with another GMP function, we add the most significant part of x to the target. If a carry is generated, we increment the exponent (unless we already had the maximum exponent, in which case we generate an overflow) and shift the result to the right; a bit is lost due to the shift but it is either the rounding bit or a following bit, and if necessary, the following bits are tested

[8] GMP now allows to use some bits for the carries, called *nail bits*. They are not supported yet in MPFR. One should note that contrary to integer operations, redundancy provided by nail bits would probably not be very interesting here due to the discontinuity of the rounding function.

and the rounding can be performed if they are not all equal.

Then, the non-significant bits of the target are taken into account; this occurs only if:

- the rounding bit is still unknown (otherwise these bits have already been taken into account before the shift due to the carry, as said above), and
- there are non-significant bits.

At this time, the rounding bit and the following bit may still be unknown; in this case, they will be determined as soon as possible from the trailing parts x'' and y''. The loops are performed on the increasing positions (by blocks), as mentioned at the end of Section 3; moreover the cases $f = 0$ and $f = 1$ are considered together (and not separated as in Section 3).

The iterations depend on the current status concerning x and y. Here are the different cases that may arise during the iterations:

- x'' has not entirely been read and y'' has not been read yet.
- x'' and y'' overlap.
- x'' has not entirely been read and y'' has entirely been read.
- x'' has entirely been read and y'' has not been read yet.
- x'' has entirely been read and y'' has not entirely been read.
- x'' and y'' have entirely been read.

In the overlapping case, at each iteration, a limb of x and the corresponding limb of y (built from two different limbs if y must be shifted) are added. The possible carry is taken into account, and the loop ends as soon as the result is 0 (all its bits are 0) for $f = 0$ or the maximum limb value `MP_LIMB_T_MAX` (all its bits are 1) for $f = 1$.

We have focused on the differences coming from the computations by blocks. The whole details may be found in the MPFR code.

5 Conclusion

We have presented the generic multiple-precision floating-point addition with exact rounding, as specified in the MPFR library, first in a rather theoretical point of view, then considering the current implementation in MPFR. The theoretical analysis could give a more regular description of the implementation, by ignoring the fact that bits are grouped into words in a computer memory. It could help to improve the current implementation (the fact that the cases $f = 0$ and $f = 1$ are considered together is probably not a very good idea, though it reduces the risk of forgetting particular cases).

The subtraction could be dealt with in a similar manner, in future work. This is a bit more complicated due to a possible cancellation (when subtracting very close numbers).

Full mechanically-checked proofs could also be considered, using the theoretical analysis to define the main notions.

References

[1] IEEE standard for binary floating-point arithmetic, Tech. Rep. ANSI-IEEE Standard 754-1985, New York, approved March 21, 1985: IEEE Standards Board, approved July 26, 1985: American National Standards Institute, 18 pages (1985).

[2] A. Feldstein, R. Goodman, Convergence estimates for the distribution of trailing digits, Journal of the ACM 23 (2) (1976) 287–297. http://portal.acm.org/citation.cfm?id=321948

[3] V. Lefèvre, J.-M. Muller, A. Tisserand, Towards correctly rounded transcendentals, IEEE Transactions on Computers 47 (11) (1998) 1235–1243. http://perso.ens-lyon.fr/jean-michel.muller/Nov98.pdf